On the Uniformization of Complements of Discriminant Loci (超函数と線型微分方程式 V)

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ON THE UNIFORMIZATION OF COMPLEMENTS OF DISCRIMINANT LOCI *)

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Introduction

Let $F(z) = 4z^3-g_2z-g_3$ be a cubic polynomial and $\Delta = 27g_3^2-g_2^3$ be its discriminant. Let us consider elliptic integrals of first and second kind:

$$I_1(g_2,g_3) = \oint F^{-\frac{1}{2}}\,dz$$

$$I_2(g_2,g_3) = \oint F^{-\frac{1}{2}}\cdot z\,dz\ ,$$

where these integrals are integrals over a homology cycle on the elliptic curve defined by $w^2 = F(z)$.

Then we can show that they satisfy the following total differential equation:

$$\begin{pmatrix} dI_1 \\ dI_2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{12}\frac{d\Delta}{\Delta} & \omega \\ -\frac{g_2}{12}\omega & \frac{1}{12}\frac{d\Delta}{\Delta} \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \end{pmatrix},$$

where $\omega = \dfrac{-3g_2dg_3 + \frac{9}{2}g_3dg_2}{\Delta}$.

Since $I_2 = (6g_3\frac{\partial}{\partial g_2} + \frac{1}{3}g_2^2\frac{\partial}{\partial g_3})I_1$, this equation reduces to an equation for I_1 of order two, which is a hypergeometric differential equation.

On the other hand, due to the theory of periods of elliptic curves, we know that two suitable linearly independent solutions $I_1^{(1)}$, $I_1^{(2)}$ give a uniformization of the complement

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of the discriminant locus of Λ to the domain $H = \{(u,v) \in \mathbb{C}^2 : I_m(u/v) > 0\}$.

In this note we want to study the problem of uniformization of complements of discriminant loci of higher dimension. For this purpose we give a natural interpretation of the above total differential equation as an equation with logarithmic poles.

In §1 we define the notion of logarithmic poles and study some basic properties of it. In particular we shall study some relationships between the topology of the complement of a divisor and algebraic properties of differential forms with logarithmic poles along the divisor.

In §2 we give a systematic method to construct systems of uniformization equations. Then an inversion problem is formulated.

In §3 we study carefully an example of type A_3, where the inversion problem is partially solved.

The complete version with proofs will be published elsewhere.

§1

First we define differential forms with logarthmic poles and state some simple properties of these forms.

Let X be an n-dimensional complex manifold and D a divisor in X. $\mathfrak{J}_D$ is the ideal for D, which is reduced and locally principal. Then we define:

$\Omega_X^p(\log D) := \{\omega$: germ of meromorphic p-form on X, ω and $d\omega$ have only simple poles along $D\}$

$Der_X(\log D) := \{\delta$: germ of holomorphic vector field on X, $\delta \mathfrak{J}_D \subset \mathfrak{J}_D\}$

$(\Omega^*_X(\log D), d)$ defines a complex of differential forms with logarithmic poles and $Der_X(\log D)$ is closed under the bracket product.

The usual pairing between forms and vectorfields can be extended to a pairing

$$\Omega^1_X(\log D) \times Der_X(\log D) \to \mathcal{O}_X$$

so that each one is a dual $\mathcal{O}_X$-module of the other. Therefore $\Omega^1_X(\log D)$ and $Der_X(\log D)$ are reflexive modules.

Let $h = 0$ be a local equation for D at $x \in D$. Then for a meromorphic p-form ω at x, the following statements are equivalent.

i) $\omega \in \Omega^p_{X,x}(\log D)$

ii) $\frac{\partial h}{\partial x_i} \cdot \omega = a \frac{dh}{h} + b$ with $a \in \Omega^{p-1}_{X,x}$, $b \in \Omega^p_{X,x}$

for $i = 1,2,\ldots,n$.

Now we define res: $\Omega^1_X(\log D) \to \mathcal{M}_{\tilde{D}}$, where $\mathcal{M}_{\tilde{D}}$ is the sheaf of mermorphic functions on the normalization of D, as follows.

$$\text{res}\ (\omega) = a/\frac{\partial h}{\partial x_i}\Big|_D$$

It can be easily checked that res is a well defined map and we have an exact sequence:

$$0 \to \Omega^1_X \to \Omega^1_X(\log D) \to \mathcal{M}_{\tilde{D}} \quad .$$

We can also show that the image of Ω^1_X (log D) under res contains the sheaf of weakly holomorphic functions $\mathcal{O}_{\ddot{D}}$.

Now we want to study the relationship between the homotopy type of the complement of D in X on one hand, and the singularities of D and properties of $(\Omega_X^*(\log D), d)$ on the other hand.

The following is one of the first results in this direction.

Theorem Let $D = D_1 \cup \cdots \cup D_m$ be the decomposition of D into irreducible components at $x \in D$, and $h = h_1 \cdots h_m$ be the corresponding prime decomposition.

Then the following statements are equivalent.

i) $\Omega^1_{X,x}(\log D) = \Omega^1_X + \sum_{i=1}^m \mathcal{O}_{X,x} \frac{dh_i}{h_i}$

ii) $\operatorname{res} \Omega^1_{X,x}(\log D) = \bigoplus_{i=1}^m \mathcal{O}_{D_{i,x}}$

iii) $\Omega^1_{X,x}(\log D)$ is generated as $\mathcal{O}_{X,x}$-module by closed forms.

iv) a) D_i is normal $i = 1,\ldots,m$.

b) There exist analytic sets $A_{ij} \subset D_i \cap D_j$ so that $\dim A_{ij} \leq n-3$ and D_i and D_j have normal crossing outside of A_{ij} for $i,j = 1,\ldots,m$

c) $\dim D_i \cap D_j \cap D_k \leq n-3$ for $i,j,k = 1,\ldots,m$.

Remark Under the conditions of the theorem we have:

$$H^1(\Omega^*_{X,x}(\log D)) \cong \mathbb{C}^m \text{ and } \pi_1^x\,(X-D) \cong \mathbb{Z}^m .$$

We can choose a system of generators $\gamma_1,\ldots,\gamma_m$ for the local fundamental group of X-D at $x \in D \subset X$, so that the usual de-Rham duality

$$\frac{1}{2\pi\sqrt{-1}} \int_{\gamma_i} \frac{dh_j}{h_j} = \delta_{ij} \text{ holds.}$$

This observation suggests that in general there exists a kind of "duality" between $\Omega_X^1(\log D)$ and $\pi_1(X-D)$.

To support this idea, let us make some observations for the case of plane curves.

i) Let D be a germ of an irreducible curve at 0 in $\mathbb{C}^2$ and B be a small ball in $\mathbb{C}^2$ centered at 0. Then $\pi_1(B-D)$ is isomorphic to the knot group $\pi_1(\partial B-D)$, which is abelian ($\cong \mathbb{Z}$) if and only if D is nonsingular at 0 (Lê). On the other hand for singular D using the Puiseux expansion we can construct forms $\omega \in \Omega_B^1(\log D)$ whose residues are not holomorphic on $\tilde{D}$.

ii) Let $D = D_1 \cup D_2$ be a germ of curve in $B \subset \mathbb{C}^2$ where D_1 and D_2 are smooth and let $m \geq 1$ be the intersection multiplicity of D_1 and D_2 at 0. For suitable coordinates $D = \{x(x-y^m) = 0\}$. Then $\pi_1(B-D)$ is a group with generators g_1, g_2 and a relation $(g_1g_2)^m = (g_2g_1)^m$ which is abelian iff $m = 1$. On the other hand $\omega = \frac{ydx - mxdy}{x(x-ym)} \in \Omega_B^1(\log D)$ has a holomorphic residue iff $m = 1$.

iii) Let $D = D_1 \cup D_2 \cup D_3$ be a germ of plane curve at $0 \in \mathbb{C}^2$ given by $xy(x-y) = 0$. Then $\pi_1(B-D)$ is a group with generators g_1, g_2, g_3 and relations $g_1g_2g_3 = g_2g_3g_1 = g_3g_1g_2$. This group is non-abelian and on the other hand $\omega = \frac{1}{x-y}\left(\frac{dx}{x} - \frac{dy}{y}\right) \in \Omega_B^1(\log D)$ has a non-holomorphic residue.

Summarizing it is easy to see that we get the following theorem.

<u>Theorem</u> Let D be a germ of curve in $\mathbb{C}^2$ at 0 and B be a small ball centered at 0 in $\mathbb{C}^2$. Then $\pi_1(B-D)$ is abelian iff

$\Omega^1_B(\log D)$ has only holomorphic residues.

Now let us come back to general situation

Theorem Let X be a complex manifold of dimension $n \geq 3$ and D be a divisor on X. Then the statements i) and ii) below are equivalent and iii) implies i) and ii).

i) $\operatorname{res} \Omega^1_X(\log D) = \mathcal{O}_{\tilde{D}}$

ii) There exists a $n - 3$ dimensional analytic set $A \subset D$, such that the singularities of $D-A$ are only normal crossings.

iii) For any point $x \in D$, the local fundamental group $\pi_1^x(X-D)$ is abelian.

Conjecture The above three statements are equivalent.

Due to a Lefschetz type theorem of Hamm-Lê, we can reduce the conjecture to the following:

Conjecture Let D be a germ of surface in $\mathbb{C}^3$ at 0 and B be a small ball in $\mathbb{C}^3$ centered at 0. Suppose the singularities of $D-\{0\}$ are only normal crossings. Then $\pi_1(B-D)$ is abelian.

We remark that the above conjecture implies a well known conjecture of Zariski: The fundamental group of the complement of a plane curve in $\mathbb{P}^2$ with only normal crossings is abelian.

In order to study our uniformization-problem we need another property of $\Omega^1_X(\log D)$.

Prop. The following statements are equivalent.

i) $\Omega^1_X(\log D)$ is a locally free $\mathcal{O}_X$-module of rank n.

ii) $\Omega^p_X(\log D) \cong {}^p\Lambda\, \Omega^1_X(\log D)$ for $p = 1,\ldots,n$.

iii) $\det \Omega^1_X(\log D) \cong \mathcal{O}_X(K_X+D)$, where K_X is the canonical line bundle of X.

Conjecture The above statements are equivalent to:

iv) For any point $x \in D$ there exists a small neighborhood B_x of x in X such that $\pi_i(B_x - D) = 0$, $i = 2,3,\ldots$.

One motive for the development of the ideas exposed in this paper is that they might be a step in the proof of this conjecture.

We remark also that in the case $n = 2$ the conjecture is true.

Example Let G be a finite reflection group acting on a complex vector space V and H be the union of hypersurfaces of reflections of G. Then the quotient space V_G can be considered as a complex affine space and $D_G = H/G$ defines a weighted homogeneous hypersurface in V_G. It was shown by E. Brieskorn and P. Deligne that the space $V_G - D_G$ is a $K(\pi,1)$. In this case we can also show, that $\Omega^1_{V_G}(\log D_G)$ is a free $\mathcal{O}_{V_G}$-module.

Now let us suppose $\Omega^1_X(\log D)$ is locally free, let $\omega_1,\ldots,\omega_n$ be a local free basis and let $X^1,\ldots,X^n \in \mathrm{Der}_X(\log D)$ be the dual basis. Then the exterior differentiation is given by $d = \sum_{i=1}^{n} \omega_i \otimes X^i$ and $\omega_1 \wedge \cdots \wedge \omega_n = \dfrac{dx_1 \wedge \cdots \wedge dx_n}{h}$

Since $\Omega^1_X(\log D)$ is locally free, it corresponds to a vector bundle over X, which we shall denote by $T^*_X(\log D)$. The inclusion $\Omega^1_X \subset \Omega^1_X(\log D)$ corresponds to a bundle homomorphism $i\colon T^*_X \to T^*_X(\log D)$, where T^*_X is the cotangent bundle of X. Remark that outside of D, i induces a bundle isomorphism.

Let us consider $\ker i$ as a subvariety of T^*_X, which we denote by L and call the Lagrangean subvariety of logarithmic poles along D. Let us also consider coker i as a bundle over D, which we denote by R and call the residue bundle of logarithmic poles along D.

Theorem i) L is a pure n-dimensional complete intersection subvariety of T_X^* such that the restriction to L of the symplectic form defining the symplectic structure on T_X^* vanishes identically.

ii) R contains a trivial line bundle T over $\tilde{D}$, such that

$$\operatorname{supp}(R/T) = \operatorname{supp}(\operatorname{res} \Omega_X^1(\log D)/ \mathcal{O}_{\tilde{D}}) \quad .$$

§2

The purpose of this paragraph is to formulate an inversion problem. First we shall develop a systematic method for constructing systems of partial differential equations (uniformization equations) which have logarithmic singularities along D and which are parametrized by a finite dimensional variety U(X,D).

We restrict from now on to the case where $X = \mathbb{C}^n$ and D is a hypersurface defined by a weighted homogeneous polynomial h with $h(t^{m_1}x_1,\dots,t^{m_n}) = t^d h(x_1,\dots,x_n)$.

We start from the assumption that $\Omega_X^1(\log D)$ is free. Let $\omega_1,\dots,\omega_n \in \Gamma(\Omega^1_{\mathbb{C}}n(\log D))$ and $X^1,\dots,X^n \in \Gamma(\mathrm{Der}_{\mathbb{C}}n(\log D))$ be global dual bases. We may choose ω_i and X^j to be homogeneous and $\omega_1 = \frac{dh}{h}$,

$$X^1 = \frac{1}{d}\sum_{i=1}^{n} m_i x_i \frac{\partial}{\partial x_i} \ .$$

Put deg $X^i = -\deg \omega_i = d_i$ and $0 = d_1 \le d_2 \le \cdots \le d_n$.

Then we have $\sum_{i=1}^{n} d_i + n = d$.

Definition Let $\mathcal{H}$ be a free $\mathcal{O}_{\mathbb{C}^n}$-module. Then a connection ∇ on $\mathcal{H}$ with logarithmic poles along D is a morphism

$$\nabla\colon \mathcal{H} \to \Omega^1_{\mathbb{C}^n}(\log D) \otimes \mathcal{H}$$

with i) $\nabla(\omega + \omega') = \nabla\omega + \nabla\omega'$

ii) $\nabla(f\omega) = df \otimes \omega + f\nabla\omega$

∇ is integrable, if the following composition is zero.

$$\mathcal{H} \xrightarrow{\nabla} \Omega^1_{\mathbb{C}^n}(\log D) \otimes \mathcal{H} \xrightarrow{\nabla} \Omega^2_{\mathbb{C}^n}(\log D) \otimes \mathcal{H}$$

∇ is homogeneous, if it is a homogeneous morphism with respect to the canonical graduation.

A connection ∇ on $\Omega^1_{\mathbb{C}^n}(\log D)$ with logarithmic poles is torsion free, if the composition

$$\Omega^1_{\mathbb{C}^n}(\log D) \xrightarrow{\nabla} \Omega^1_{\mathbb{C}^n}(\log D) \otimes \Omega^1_{\mathbb{C}^n}(\log D) \to \overset{2}{\Lambda}\Omega^1_{\mathbb{C}^n}(\log D)$$

coincides with the exterior differentiation d.

Let us denote by $U(\mathbb{C}^n, D)$ the set of all integrable, torsion free, homogenous connections with logarithmic poles along D. The first remark we make is that $U(\mathbb{C}^n, D)$ has the structure of a finite dimensional algebraic variety.

Conjecture $U(\mathbb{C}^n, D) \neq \emptyset$

Example Let G, V_G, D_G be as in example in §1.

Then for $G = A_k, D_k, E_k$, $U(V_G, D_G) \neq \emptyset$.

Now using the bases ω_i, X^j, we determine the coefficients $\omega_i^j \in \Gamma(\Omega^1_{\mathbb{C}^n}(\log D))$ and $\Gamma_i^{jk} \in \Gamma(\mathcal{O}_{\mathbb{C}^n})$ for $\nabla \in U(\mathbb{C}^n, D)$.

$$\nabla\omega_i = \sum_{j=1}^{n} \omega_i^j \otimes \omega_j \ , \ \omega_i^j = \sum_{k=1}^{n} \Gamma_i^{jk}\, \omega_k \ .$$

Then ∇ is integrable $\Leftrightarrow$ $d\,\omega_i^j = \sum_{k=1}^{n} \omega_i^k \wedge \omega_k^j$

∇ is torsion free $\Leftrightarrow$ $d\,\omega_i = \sum_{j=1}^{n} \omega_i^j \wedge \omega_j$

∇ is homogeneous ω_i^j is homogenous of degree $d_j - d_i$.

<u>Prop.</u> i) An element $\omega \in \Omega^1_{\mathbb{C}^n}(\log D)$ is horizontal (i.e., $\nabla\omega = 0$) iff there exists a function u such that

a) $\omega = du = \sum_{i=1}^{n} X^i u \cdot \omega_i$

b) $X^k X^j u + \sum_{i=1}^{n} X^i u\, \Gamma_i^{jk} = 0$ for $k, j = 1, \ldots, n$.

Let us call the system of equations (b) the system of uniformization equations with respect to $\nabla \in U(\mathbb{C}^n, D)$.

ii) The system of uniformization equations has $n + 1$ $\mathbb{C}$-linearly independent solutions, which are multivalued holomorphic functions on $\mathbb{C}^n - D$.

We remark that the system of uniformization equations is a maximally overdetermined system in the sense of the theory of partial differential equations. To show this, we have only to check that the symbol ideal of the system is nothing but $(\sigma(X^k)\sigma(X^j),\ k,j = 1,\ldots,n) = (\sigma(X^i),\ i = 1,\ldots,n)^2$ whose zero locus is just the Lagrangean variety L of logarithmic poles along D.

In order to obtain some general properties of the solutions of the uniformization equations, we give some simple results on the coefficients $\Omega = (\omega_i^j)$ and $\Gamma = \Gamma_i^{jk}$.

<u>Prop</u>. For a given $\nabla \in U(\mathbb{C}^n, D)$ there exists a constant $s \in \mathbb{C}$ so that $\Gamma_j^{i1} = \frac{s+d_i}{d}\,\delta_j^i$, for $i,j = 1,\ldots,n$ and

$\Gamma_j^{1i} = \frac{s}{d}\,\delta_j^i$ (i.e., $\omega_i^1 = \frac{s}{d}\,\omega_i$) for $i,j = 1,\ldots,n$.

<u>Cor</u>. We can choose $1, u_1,\ldots,u_n$ as a base system for the solutions of the uniformization equations, where the u_i are homogeneous of degree s (i.e., $X^1 u_i = \frac{s}{d}\,u_i$ or symbolically

$$u_i(t^{m_1}x_1,\ldots,t^{m_n}x_n) = t^s\,u_i(x_1,\ldots,x_n)).$$

Now since $\operatorname{tr}\Omega = \sum_{i=1}^{n} \omega_i^i$ is a closed form it has a presentation $\sum_{i=1}^{m} s_i \frac{dh_i}{h_i}$ $\left(= \left(\frac{ns - \sum_{i=1}^{m} m_i}{d} + 1\right)\frac{dh}{h}\right.$ in the case when h is irreducible) where the h_i are the irreducible components of h and $s_i \in \mathbb{C}$.

Then we easily compute:

$$\det(X^i u_j) = c \exp\int \operatorname{tr}\Omega = c h_1^{s_1} \cdots h_m^{s_m}$$

and

$$\det\frac{\partial(u_1\ldots u_n)}{\partial(x_1\ldots x_n)} = \frac{\omega_1\wedge\cdots\wedge\omega_n}{dx_1\wedge\cdots\wedge dx_n}\det(X^i u_j) = c\, h_1^{s_1-1}\cdots h_m^{s_m-1},$$

where c is a non zero constant.

Let $p\colon \widetilde{\mathbb{C}^n - D} \to \mathbb{C}^n - D$ be the universal covering of $\mathbb{C}^n - D$. Let $\bar{p}\colon \widetilde{\mathbb{P}^{n-1} - \bar{D}} \to \mathbb{P}^{n-1} - \bar{D}$ be the universal covering of $\mathbb{P}^{n-1} - \bar{D}$, in the case when h is homogeneous, where $\bar{D}$ is the hypersurface in $\mathbb{P}^{n-1}$ defined by $h = 0$.

Then (for $s \neq 0$) we obtain holomorphic a mapping $u_V = (u_1,\ldots,u_n) : \widetilde{\mathbb{C}^n-D} \to \mathbb{C}^n - \{0\}$ which commutes with the $\mathbb{C}^*$ action and which is of maximal rank. In the case when h is homogeneous we obtain also a mapping

$$\bar{u}: \widetilde{\mathbb{P}^{n-1}-\bar{D}} \to \mathbb{P}^{n-1}$$

of maximal rank so that the diagram below commutes.

$$\begin{array}{ccc} \widetilde{\mathbb{C}^n-D} & \to & \mathbb{C}^n-\{0\} \\ \downarrow & & \downarrow \\ \widetilde{\mathbb{P}^{n-1}-\bar{D}} & \to & \mathbb{P}^{n-1} \end{array}$$

Now we are in a position to state the inversion problem.

I. How do the image domains H of u and $\bar{H}$ of $\bar{u}$ look like? In particular what can be said about the boundary of H and $\bar{H}$?

II. When do the values $u_i(x)$ of the integrals determine the value x? In other words: when do the inversion mappings $E: H \to \widetilde{\mathbb{C}^n-D}$ and $\bar{E}: \bar{H} \to \widetilde{\mathbb{P}^{n-1}-\bar{D}}$ with $\bar{E} \circ \bar{u} = \bar{p}$,Eou = exist?

III. Describe the inversion map $x = E(u)$ explicitly, when it exists. Can it be developed as an Eisenstein series?

For n = 2 the inversion problem which we formulate here can be solved for certain special cases by means of the classical Schwarz Christoffel theory.

In general the problem seems delicate and requires a deeper analysis of the boundary of H and $\bar{H}$, which we don't accomplish here.

§3

In §1 we have noted that an interesting example in the context of the problem of uniformizing complements of discriminants is the case where $V_G = V/G$, G a complexified reflection group, and where the discriminant $D_G \subset V_G$ is the image of the reflection hyperplanes.

In this paragraph, we shall study in detail the case where G is the Weyl group of type A_3. We shall analyze the systems of uniformization equations for this case in analogy with the classical Schwarz Christoffel theory for the hypergeometric differential equations. In the case of the hypergeometric equation, which depends on parameters, one has an inversion problem, which is solved by the classical theory and the behavior of the different types of inversion maps is well understood.

In our case $G = A_3$ it turns out that the variety $U(V_G, D_G)$ parametrizing the family of all uniformization equations is a 1-dimensional variety with two irreducible components. At present we are not able to solve the inversion problem for all possible parameter values in $U(V_G, D_G)$. However for a certain special point in $U(V_G, D_G)$, which we call a complete elliptic point, we can solve the inversion problem for the corresponding uniformization equations. We shall describe the solutions of this uniformization equation by means of elliptic integrals. This will be possible because in the case A_3, we can describe $V_G - D_G$ as a fibration with fibres which are punctured elliptic curves. Finally the inversion $x = E(u)$ will be given by the classical Eisenstein series of Weierstrass.

This reduction to the elliptic integrals above comes from the fact that the Galois group of a polynomial of degree 4 is solvable. Hence we don't have any generalization of the above reduction process for A_n with $n \geq 4$.

Let us denote by V_3 (resp. V_2) the three (resp. two) dimensional complex affine space of all polynomials of degree 4 (resp. 3) of the form

$$F(t) = t^4 + x_2 t^2 + x_3 t + x_4$$

$$(\text{resp.} \quad F(z) = 4z^3 - g_2 z - g_3) .$$

Let D_3 (resp. D_2) be the hypersurface in V_3 (resp. V_2) which is the zero locus of the discriminant Δ_3 (resp. Δ_2) of the polynomial $F(t)$ (resp. $F(z)$), where

$$\Delta_3(x_2,x_3,x_4) = 256x_4^3 - 128x_4^2x_2^2 + x_4(16x_2^4+144x_2x_3^2) - (4x_2^3x_3^2+27x_3^4)$$

$$\Delta_2(g_2,g_3) = 27g_3^2 - g_2^3$$

Then $\Omega^1_{V_3}(\log D_3)$, $\mathrm{Der}_{V_3}(\log D_3)$ are free and dual bases are given by

$$X^1 = \frac{1}{6}\left(2x_2 \frac{\partial}{\partial x_2} + 3x_3 \frac{\partial}{\partial x_3} + 4x_4 \frac{\partial}{\partial x_4}\right)$$

$$X^2 = 3x_3 \frac{\partial}{\partial x_2} + (-x_2^2 + 4x_4) \frac{\partial}{\partial x_3} - \frac{1}{2} x_2 x_3 \frac{\partial}{\partial x_4}$$

$$X^3 = (x_2^2 - 12x_4)\frac{\partial}{\partial x_2} + 3x_2x_3\frac{\partial}{\partial x_3}$$

$$+ (\frac{9}{4}x_3^2 - 4x_2x_4)\frac{\partial}{\partial x_4}$$

$$\omega_1 = \frac{1}{2}\frac{d\Delta_3}{\Delta_3}$$

$$\omega_2 = \frac{1}{\Delta_3}((32x_2x_3x_4 - 9x_3^3)dx_2 + (-16x_2^2x_4+6x_2x_3^2+64x_4^2)dx_3$$

$$- (4x_2^2x_3 + 48x_3x_4)dx_4)$$

$$\omega_3 = \frac{2}{3}\frac{1}{\Delta_3}((8x_2^2x_4-3x_2x_3^2-32x_4^3)dx_2 + (2x_2^2x_3+24x_3x_4)dx_3$$

$$+ (-4x_2^3+16x_2x_4-18x_3^2)dx_4)$$

From this we obtain the following relations:

$$[X^1,X^i] = \frac{i-1}{6}X^i \quad i = 1,2,3 \ , \ [X^2,X^3] = x_2X^2$$

$$d\omega_1 = 0, \ d\omega_2 = -\frac{1}{6}\omega_1 \wedge \omega_2 - x_2\omega_2 \wedge \omega_3, \ d\omega_3 = -\frac{1}{3}\omega_1 \wedge \omega_3$$

$$\omega_1 \wedge \omega_2 \wedge \omega_3 = 8\frac{dx_2 \wedge dx_3 \wedge dx_4}{\Delta_3}$$

For $\Omega^1_{V_2}$ (log D_2) we have the following basis and relations:

$$\bar{\omega}_1 = \frac{d\Delta_2}{\Delta_2} \ , \ \bar{\omega}_2 = \frac{3g_3dg_2 - 2g_2dg_3}{\Delta_2}$$

$$d\bar{\omega}_1 = 0 \ , \ d\bar{\omega}_2 = \overset{-\frac{1}{6}}{\cancel{-\frac{1}{3}}}\bar{\omega}_1 \wedge \bar{\omega}_2 = \overset{1}{\cancel{\frac{1}{18}}}\frac{dg_2 \wedge dg_3}{\Delta_2}$$

We remark that ω_1 and ω_2 form an involutive system.

Theorem I. Let us put $L = 16x_4 + \frac{4}{3}x_2^2$ and $M = X^3L = \frac{8}{3}x_2^3 + 36x_3^2 - 96x_2x_4$. Then we obtain:

i) $16\Delta_3 = L^3 - \frac{1}{3} M^2$

ii) $\mathbb{C}[L,M] = \{P \in \mathbb{C}[x_2,x_3,x_4]: X^2P = 0\}$

II. Let $N: V_3 \to V_2$ be the holomorphic map defined by $g_2 = L(x_2,x_3,x_4)$ and $g_3 = -\frac{1}{9} M(x_2,x_3,x_4)$. Then we have:

i) $N^*(\Delta_2) = -16\Delta_3$

ii) $N^*(\overline{\omega}_1) = 2\omega_1$, $N^*(\overline{\omega}_2) = -\omega_3$

III. The fundamental groups of V_3-D_3 and V_2-D_2 are braid groups, which are presented by suitable generators as follows:

$\pi_1(V_3-D_3)$: generators a_1,a_2,a_3 relations $a_1a_2a_1=a_2a_1a_2$,

$a_2a_3a_2=a_3a_2a_3$

$a_1a_3=a_3a_1$

$\pi_1(V_2-D_2)$: generators b_1,b_2 relations $b_1b_2b_1=b_2b_1b_2$

such that $N_*(a_1) = b_1$, $N_*(a_2) = b_2$, $N_*(a_3) = b_1$.

IV. Let $X = \{(g_2,g_3)\times(y,z,w) \in V_2\times\mathbb{P}^2$:

$$yw^2=4z^3-g_2y^2z-g_3y^3\}$$

be the Weierstrass family of elliptic curves and D the divisor of X defined by $y = z = 0$. Then $y = 1$, $z = -\frac{2}{3} x_2$, $w = 2x_3$, $g_2 = L$, $g_3 = -\frac{1}{9} M$ defines a holomorphic map $I: V_3 \to X$ such that

i) $I: V_3 \to X-D$ is an isomorphism

ii) The diagram
$$\begin{tikzcd} V_3 \arrow[r, "I"] \arrow[dr] & X \arrow[d] \\ & V_2 \end{tikzcd}$$
is commutative.

Cor. An orbit of X^2 which is outside of D_3 is identified by I with an elliptic curve punctured at infinity.

Remark. For the space of polynomials of higher degree, there exists no fibration $V_m \to V_n$ of this type, except the above case and the trivial cases of $n = 1$.

Now let us compute $U(V_3, D_3)$.

Let Ω be the 3 x 3 matrix with coefficients in $\Omega^1_{V_3}(\log D_3)$.

$$\begin{pmatrix} g\omega_1 + ax_2\omega_3 & g\omega_2 & g\omega_3 \\ dx_2\omega_2 + ex_3\omega_3 & (g+\frac{1}{6})\omega_1+kx_2\omega_3 & f\omega_2 \\ (bx_2^2 + cx_4)\omega_3+hx_3\omega_2 & ix_2\omega_2+jx_3\omega_3 & (g+\frac{1}{3})\omega_1+\ell x_2\omega_3 \end{pmatrix}$$

where a,b,c,d,e,f,g,h,i,j,k,ℓ are undetermined constant coefficients in $\mathbb{C}$.

Applying the integrability condition $d\Omega = \Omega \wedge \Omega$, we compute the coefficients. Thus we obtain the following two components of $U(V_3, D_3)$.

Type I

$$\Omega = \left[\begin{array}{c:c:c}
(\frac{k}{3} - \frac{1}{6})\omega_1 & (k - \frac{1}{2})\omega_2 & (k - \frac{1}{2})\omega_3 \\ \hdashline
-\frac{4}{3}kx_2\omega_2 + 3kx_3\omega_3 & \frac{k}{3}\omega_1 + kx_2\omega_2 & -\frac{2}{3}k\omega_2 \\ \hdashline
((2k+\frac{1}{2})x_2^2 + 6x_4)\omega_3 & (k+1)x_2\omega_2 & (\frac{k}{3} + \frac{1}{6})\omega_1 \\
+ 3kx_3\omega_2 & -\frac{9}{4}(2k+1)x_3\omega_3 & - kx^2\omega_3
\end{array}\right]$$

$k \in \mathbb{C}$

The inditial equation is $\lambda^2(\lambda - \frac{k}{2}) = 0.$

The system of uniformization equations is

$$(X^1 + \frac{1}{6})u = \frac{k}{3}u$$

$$(X^2)^2 u = k(2x_2 X^1 - \frac{1}{3}X^3)u$$

$$X^3X^2 u = k(9x_3 X^1 + x_2X^2)u$$

$$((X^3)^2 - \frac{9}{8}LX^1 + \frac{9}{4}x_3X^2)u = k(6x_2^2X^1 - \frac{9}{2}x_3X^2 - x_2X^3)u .$$

Type II

$$\Omega = \left[\begin{array}{c:c:c}
(\frac{k}{3} - \frac{1}{6})\omega_1 & \omega_2 & \omega_3 \\ \hdashline
-\frac{1}{3k}(4k-2)x_2\omega_2 & \frac{k}{3}\omega_1 + kx_2\omega_3 & -\frac{2}{3k}\omega_2 \\
-3(\frac{k}{2}-1)x_3\omega_3 & & \\ \hdashline
(-\frac{1}{4}(k+4)(k-2)x_2^2 & (k+1)x_2\omega_2 & (\frac{k}{3}+\frac{1}{6})\omega_1 \\
+3(3k^2+2k-4)x_4)\omega_3 & -\frac{9}{4}k(k+2)x_3\omega_3 & -kx_2\omega_3 \\
-\frac{3}{2}(k-2)x_3\omega_2 & &
\end{array}\right]$$

where $k \in \mathbb{C}^*$

The inditial equation is $\lambda(\lambda-\frac{k}{3}+\frac{1}{2})(\lambda-\frac{k}{3}-\frac{1}{2}) = 0$.

The system of uniformization equations is

$$X^1 u = (\frac{k}{3} - \frac{1}{6})u$$

$$(X^2)^2 u = -\frac{2}{3k}(2k-1)x_2 u - \frac{2}{3k}X^3 u$$

$$X^3 X^2 u = -3(\frac{k}{2} - 1)x_3 u + kx_2 X^2$$

$$(X^3)^2 u = (-\frac{1}{4}(k+4)(k-2)x_2^2 + 3(3k^2+2k-4)x_4)u$$
$$-\frac{9}{4}k(k+2)x_3 X^2 u - kx_2 X^3 u$$

Remark i) The above two types of systems give the same equations when $k = \pm 1$.

ii) The solutions of the system of Type II do not have an Euler integral presentation, since the initial equation has separated roots.

Now let us consider the Type I. Since the characteristic exponents of the equations are $0, 0, \frac{k}{2}$, it is necessary for the existence of an inversion may that $\frac{k}{2} = \frac{1}{m}$ for some integer m. The number m is the order of the semisimple part of the monodromy transformation obtained by walking once around a general point of the discriminant surface D_3.

In the particular case when $k = 1$ a basis of the linear system of solutions of the uniformization equations of Type I gives just the algebraic functions which are the components of the inverse "map" of the ramification covering

$V_{G_{\Lambda_3}} \to V_{\Lambda_3}$. The fact that these algebraic functions can be developed as hypergeometric series of several variables was known to Mellin.

Now let us study the most degenerate case, when $m = \infty$, i.e., the case $k = 0$. In this case from the equations we check easily that X^2u must be a constant. From this one can deduce that the solutions of the uniformization equations can be obtained as solutions of one of the two following systems:

i) $X^1u + \frac{1}{6}u = 0$

$X^2u = 0$

$((X^3)^2 + \frac{3}{16} L)u = 0$

ii) $X^1u + \frac{1}{6}u = 0$

$X^2u = -1$

$((X^3)^2 + \frac{3}{16}L)u = \frac{9}{4}x_3$

<u>Theorem</u> Let us put

$$u_i(x) = \oint_{\gamma_i(x)} \frac{dz}{\sqrt{4z^3 - Lz + \frac{1}{9} M}} \qquad i = 1,2.$$

$$v(x) = \int_{\infty}^{-\frac{2}{3}x_2} \frac{dz}{\sqrt{4z^3 - Lz + \frac{1}{9} M}} ,$$

where $\gamma_i(x)$ $i = 1,2$ is a horizontal family of canonical bases of the first homology group of the elliptic curves $W^2 = 4z^3 - L(x)z + \frac{1}{9} M(x)$.

Then we obtain:

i) $u_i(x)$ $i = 1,2$ satisfies the system of equations i).

$v(x)$ satisfies the system of equations ii).

ii) Let $X = \{(u_1,u_2,v) \in \mathbb{C}^3 : \mathrm{Im}(u_1/u_2) > 0 \ v \neq mu_1 + nu_2$ for $m,n \in \mathbb{Z}\}$

Then the image domain of the multivalued map $(u_1(x), u_2(x), v(x)): V_3 - D_3 \to \mathbb{C}^3$ is equal to X.

iii) The following functions defined on X give the inversion map for $(u_1,u_2,v): V_3-D_3 \to X$.

$$-\frac{2}{3}x_2 = p(v,u_1,u_2) = v^{-2} + \sum{}'\{(v-nu_1-mu_2)^{-2} - (nu_1+mu_2)^{-2}\}$$

$$2x_3 = p'(v,u_1,u_2) = -2\left\{v^{-3} + \sum{}'(v-nu_1-mu_2)^{-3}\right\} \qquad E_v$$

$$L = 16x_4 + \frac{4}{3}x_2^2 = 60 \sum{}' (nu_1 + mu_2)^{-4} \qquad E_{u_1}$$

$$-\frac{1}{9}M = -\frac{8}{27}x_2^3 - 4x_3^2 + \frac{32}{3} x_2x_4 = 140 \sum{}'(nu_1+mu_2)^{-6}.$$

Here $\sum'$ means the summation for all $m,n \in \mathbb{Z}$ except $(m,n) = (0,0)$.

For the proof we use the following lemma.

<u>Lemma</u> Let $P(z) = 4z^3 - L(x)z + \frac{1}{9}M(x)$. Then

$$((X^3)^2 + \frac{3}{16} L)\frac{1}{\sqrt{P}} = \frac{\partial}{\partial z} \frac{-12Lz^4 - 3L^2z^2 + \frac{2}{3} MLz - \frac{M^2}{3} + L^3}{8P\sqrt{P}}$$

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